

Ques - Construct an example of discontinuous linear transformation.

Solⁿ - Let M be a linear subspace of $C[0,1]$, where M consists of all continuous and differentiable functions $f(s)$ on $[0,1]$ such that $f'(s)$ continuous on $[0,1]$ consider the linear transformation

$T: M \rightarrow C[0,1]$ defined by

$$Tf = \frac{d}{ds} (f(s))$$

We show that T is not bounded

$$\text{Let } f_n(s) = \sin n\pi s, \text{ then } Tf_n = \frac{d}{ds} (f_n(s)) \\ = n\pi \cos n\pi s$$

$$\Rightarrow \|f_n\| = \sup_{0 \leq s \leq 1} |\sin n\pi s| = 1$$

$$\Rightarrow \|Tf_n\| = \sup_{0 \leq s \leq 1} |n\pi \cos n\pi s| = n\pi \text{ for } n=1,2,3,\dots$$

If possible, suppose there exists $M > 0$ such that

$$\|Tf\| \leq M \|f\| \text{ for all every } f \in M$$

$$\|Tf_n\| \leq M \|f_n\| \text{ i.e. } n\pi \leq M$$

for $n=1,2,3,\dots$ which is clearly not true.

Hence T is not a bounded linear transformation.

